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Statistical energy analysis SEA formulates the dynamic response of a system in terms of power and energy variables. The SEA parameters include power inputs; damping loss factors; which control the power dissipated within the system; and coupling loss factors, which control the power transmitted between coupled subsystems. One of the great difficulties in using SEA is the calculation of these parameters. In this paper SEA parameters are formulated using general mobility functions. Simplifications that result from averaging the parameters either over frequency or over an ensemble of dynamic systems are presented. These simplifications make it possible to apply SEA to very complex structural-acoustic systems.

1. Introduction

Statistical energy analysis, or SEA as it is commonly called, provides a technique to study the dynamic response of complex structural-acoustic systems. Since its introduction almost 30 years ago, SEA has slowly grown in popularity. Today it is considered by many to be the best analysis technique for high frequencies (Rockwood 1987). In addition, research is being conducted at Cambridge Collaborative and other organizations to extend the validity of the technique to lower frequencies (Manning 1990). The rise in popularity of SEA is due to the fact that it can be used both as a theoretical technique and an experimental technique. The theoretician computes the SEA parameters using theoretical mode and wave analysis, while the experimentalist computes these parameters using various measured data. While both techniques have merit, this paper focuses more on the theoretical side of SEA than the experimental. It is hoped that those trying to compute SEA parameters from basic theory will benefit from the approaches outlined in the paper and that those trying to measure SEA parameters will be able to design better and more accurate measurement techniques given a better understanding of the theory.

In several past papers an effort has been made to verify the validity of SEA concepts (Hodges 1986; Langley 1990). These are important studies. However, in this paper I emphasize more the engineering development of SEA models using theoretically computed SEA parameters based on mobility and impedance functions.

The paper is divided into two major sections. The first section deals with the general process of calculating the SEA parameters using mobility functions. In this section, I assume that the power input to a system or the power transmitted between systems can be represented by the product of single force and velocity amplitudes. This would be the case for a point excitation in a single direction or a point connection between systems at which all but one of the degrees of freedom are constrained. In the second major section of the paper the formulations for the SEA

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parameters are expanded to deal with excitations and connections that include multiple degrees of freedom and may be extended over a line or area. The emphasis is on a practical approach toward computing coupling loss factors.

2. SEA parameters

In SEA the equations of motion describing a dynamic system are cast in terms of power and energy variables (Lyon 1975). Power balance equations are developed by requiring the overall time-average power input to a system to be equal to the sum of the time-average power dissipated within the system due to damping and the net time-average power transmitted to other systems. The power balance equations are appealing because of their simplicity. The difficulties arise in calculating the parameters that govern the power input and the power transmitted. In the following sections these parameters will be formulated using a general mobility formulation.

(a) *Power input*

The power inputs to the SEA subsystems provide the basic forcing functions for the power balance equations. In some cases the power input to a system can be measured or determined empirically. However, because of the difficulty in obtaining a direct measurement of power, a calculation of power input using mobility functions – and their inverse, impedance functions – is often required. The mobility functions can be predicted analytically and in many cases measured mobility functions can be used to improve the confidence in the SEA model. It may also be possible to compute the power input to a dynamic system using a finite element model. The finite element prediction of the power input can then be combined with an SEA model of the structure and acoustic spaces to provide a composite model that combines the advantages of the finite element and the statistical energy analysis methods.

In the following section the time-average power input to mechanical and acoustic systems will be formulated for general point, line, and area excitation sources. First, however, we introduce the basic concepts of mobility analysis. For this introduction we assume that the force and velocity of the system can be represented by single variables, F and V.

The mobility function for a mechanical system is the ratio of the complex amplitude of the response velocity to the complex amplitude of the force acting on the system, where harmonic $e^{+j\omega t}$ time-dependence is assumed. In general, the mobility function depends on frequency. The real part of the mobility is the conductance, which is always positive. The imaginary part of the mobility is the susceptance, which can be negative or positive. A change in the sign convention for the complex time dependence, e.g. from $\exp^{+j\omega t}$ to $\exp^{-j\omega t}$, changes the sign of the imaginary part of the mobility but not the real part.

The power input to a system is the product of force and velocity. The time-average power input can be expressed in terms of the complex amplitudes of the force and velocity as

$$W^{\mathrm{in}} = \tfrac{1}{2}\,\mathrm{Re}(F^{*}V), \tag{1}$$

where W^{in} is the time-average power input, Re signifies the 'real part of', V is the velocity amplitude, and F^{*} is the complex conjugate of the force amplitude. If the amplitude of the force acting on the structure is known, the power input can be expressed in terms of the magnitude-squared of the force amplitude and the real part

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of the mechanical mobility; the conductance, G. The mean-square force acting on the structure, $\langle F^2 \rangle$, is equal to one-half the magnitude-squared of the force amplitude so that the power input can be written in terms of the conductance as

$$W^{\text{in}} = \langle F^2 \rangle G. \tag{2}$$

The formulation of the power input in terms of the forces acting on the system and the conductance is generally useful when the forces are known or can be measured. In other cases the excitation force is not known. However, the excitation can be defined by an imposed velocity. In this case the power input can be expressed in terms of the mean-square velocity, $\langle V^2 \rangle$, and the real part of the mechanical impedance, which is generally referred to a the resistance, R,

$$W^{\text{in}} = \langle V^2 \rangle R. \tag{3}$$

In the most general case, an interaction exists between the excitation and the structure so that neither the excitation force nor an imposed velocity can be directly specified. In this case an equivalent mechanical 'circuit' can be set up to model the excitation (Shearer 1971). The concepts of a Thevenin-equivalent and a Norton-equivalent circuit are borrowed from electrical engineering. The Thevinen equivalent circuit can be used to represent a general source of excitation in terms of a source impedance and its 'blocked-force'; the force applied to the structure if it were rigidly constrained from moving. The Norton equivalent circuit can be used to represent the same excitation source in terms of a source impedance and its 'free-velocity'; the velocity of the source when it is detached from the structure.

By using the Thevinen or blocked-force representation of the excitation, the power input to the structure can be written in terms of the mean-square blocked-force as

$$W^{\text{in}} = \langle F_{\text{b}}^2 \rangle R / |Z_{\text{s}} + Z|^2, \tag{4a}$$

where $\langle F_{\text{b}}^2 \rangle$ is the mean-square blocked force, Z_{s} is the source impedance of the excitation, and Z is the impedance of the excited structure. The power input from the same excitation source can also be written using the Norton or free-velocity representation in terms of the mean-square free-velocity as,

$$W^{\text{in}} = \langle V_{\text{f}}^2 \rangle G / |Y_{\text{s}} + Y|^2, \tag{4b}$$

where $\langle V_{\text{f}}^2 \rangle$ is the mean-square free velocity and Y_{s} is the source mobility (inverse of impedance). As the power input determined from (4a, b) must be equal we find that the mean-square blocked force and free velocity are related by the magnitude squared of the source impedance,

$$\langle F_{\text{b}}^2 \rangle / \langle V_{\text{f}}^2 \rangle = |Z_{\text{s}}|^2. \tag{4c}$$

This relation can be used as a means to determine the source impedance from measured data or to relate the blocked force and the free velocity in formulations of power input.

The general mobility formulation can be extended to the case where an excitation is applied to the junction of several systems. For this case a junction impedance is defined. Because the systems connected at a structural junction share a common velocity at the junction the junction impedance is the sum of the source impedance plus the impedances of all systems connected to the junction,

$$Z_{\text{jnc}} = Z_{\text{s}} + \sum_i Z_i, \tag{5a}$$

where the summation is over all systems connected at the junction. The power input to one of the connected subsystems can be written in terms of the mean-square blocked force and the junction impedance as

$$W_{\mathrm{r}}^{\mathrm{in}} = \langle F_{\mathrm{b}}^2 \rangle R_{\mathrm{r}} / |Z_{\mathrm{jnc}}|^2, \tag{5b}$$

where $W_{\mathrm{r}}^{\mathrm{in}}$ is the power input to the receiving system r and R_{r} is the resistance of system r. The formulation in terms of free velocity and system mobilities is more complex since the mobility of a structural junction is not the sum of the mobilities of the connected systems.

(*b*) *Random excitation*

When the excitation force is a random function of time, the power input in a band of frequencies can be found by integrating the product of the spectral density of the force and the conductance over the band, $\Delta\omega$,

$$W^{\mathrm{in}} = \int_{\Delta\omega} \mathrm{d}\omega \, S_{\mathrm{F}}(\omega) \, G(\omega), \tag{6a}$$

where ω is the radian frequency, W^{in} is the power input in the frequency band, $\Delta\omega$, and $S_{\mathrm{F}}(\omega)$ is the power spectral density of the excitation force. For many cases of practical interest the spectral density of the force can be assumed to be fairly constant over the frequency band $\Delta\omega$. Then the power input can be rewritten in terms of the average conductance and the mean-square force within the band,

$$W^{\mathrm{in}} = \langle F^2 \rangle \langle G \rangle_{\Delta\omega}, \tag{6b}$$

where $\langle\,\rangle_{\Delta\omega}$ signifies an average over the frequency band $\Delta\omega$. The process of averaging over the frequency bandwidth can be easily extended to a velocity source by using a frequency-averaged mechanical resistance in (3).

The use of a frequency-average impedance or mobility function to evaluate the power input for the general excitation represented by a blocked force or a free velocity and a source impedance is not immediately clear. As in the earlier derivation for the force and velocity sources, we should average the input power given by (4) or (5) over a band of frequencies. Although formally correct this averaging process brings about no immediate simplification. It is common in most SEA formulations to use average impedance or average mobilities to evaluate the power input using (4) or (6),

$$W^{\mathrm{in}} = \langle F_{\mathrm{b1}}^2 \rangle \frac{\langle R \rangle}{|Z_{\mathrm{s}} + \langle Z \rangle|^2} \quad \text{or} \quad W^{\mathrm{in}} = \langle V_{\mathrm{fr}}^2 \rangle \frac{\langle G \rangle}{|Y_{\mathrm{s}} + \langle Y \rangle|^2}, \tag{7}$$

where $\langle\,\rangle$ signifies an average over the frequency band. The validity of using an average impedance or mobility to determine the power input is a key question in assessing the accuracy of many SEA parameter formulations. As we will see in a later section the same question arises when computing the coupling loss factors, because an impedance or mobility formulation of the power transmission is generally used.

At very high frequencies, where the average spacing between resonance frequencies is small compared with the response bandwidth of the individual resonances, the mechanical impedance functions will be fairly smooth functions of frequency. In this case the two results given above will be the same, so that using the average impedance or mobility to evaluate the power input is valid.

The use of an average impedance or mobility is also valid if the excitation source impedance is either much greater than or much smaller than the mechanical

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impedance of the structure. In such cases the source can be modelled either as a velocity or force source and the power input can be evaluated using the average resistance or conductance.

At lower frequencies and for very lightly damped structures, the impedance functions can vary significantly over a band of frequency. At these frequencies the use of the average impedance or mobility to evaluate the power input may result in a biased estimate. Further work in this area is warranted.

(*c*) *Statistical estimates of the mobility functions*

The use of a frequency-average conductance for a random excitation is a common example where a statistical estimate of a mobility function is used to determine the power input. However, a broader application of the average conductance can be made which is applicable both to random and deterministic single frequency excitations.

In SEA, the structure is described statistically so that the resonance frequencies and mode shapes become random variables. In this case we can define an ensemble of structures and use ensemble averages and other statistical measures to define the structural parameters. Thus, the power input for a force source averaged over the ensemble can be written in terms of an ensemble average conductance,

$$\langle W^{\mathrm{in}} \rangle_{\mathrm{ens}} = \langle F^2 \rangle \langle G(\omega) \rangle_{\mathrm{ens}}, \tag{8}$$

where $\langle G(\omega) \rangle_{\mathrm{ens}}$ is the ensemble-average conductance at the single frequency, ω.

The concept of an ensemble of structures may be difficult to accept for many engineers. It can imply poor manufacturing tolerances and poor quality so that the actual dimensions of the structure vary significantly for the different members of the ensemble. This need not be the case, particularly at frequencies above the first few system resonance frequencies. An ensemble can be formed from identical structures with small changes in operating conditions, temperature, and other variables that cause a seemingly random variation of resonance frequencies and mode shapes. Also during the preliminary design of a product, an ensemble of systems can be formed with variations to account for design uncertainties. As the design matures these uncertainties can be removed and the prediction uncertainties can be removed.

The definition of an ensemble of structures can also be applied when the excitation force is random. In this case the assumption that the force spectral density can be constant over the frequency band need not be made and the ensemble-average conductance can be used to relate the spectral densities of the power input and the force on a narrow-band basis,

$$\langle S_W(\omega) \rangle_{\mathrm{ens}} = S_{\mathrm{F}}(\omega) \langle G(\omega) \rangle_{\mathrm{ens}}, \tag{9}$$

where $S_W(\omega)$ is the spectral density of the power input; formally, the real part of the cross-spectrum of the force and velocity.

The frequency or ensemble average mobility of a system can be formulated in terms of the modes of the system. Following a classical modal analysis the response of the system can be expressed as a sum of the responses of the individual modes,

$$V(x) = \sum_{n=1}^{\infty} V_n \psi_n(x), \tag{10}$$

where V_n is the complex amplitude of the response of the nth mode and $\psi_n(x)$ is the mode shape of the nth mode. The response amplitude, V_n, can be found as the product

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of the force acting on the mode and the mobility for a single degree-of-freedom system. For a point force acting on the system at point x the modal force amplitude is the product of the applied force amplitude and the value of the mode shape at the application point of the force. The mobility function for a single mode of the system represented by a mass/spring oscillator can be written in terms of the mass, the resonance frequency, and the damping loss factor,

$$Y_n(\omega) = \frac{1}{M_n} \frac{\omega^2 \omega_n \eta_n + j\omega(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (\eta_n \omega_n \omega)^2}, \tag{11a}$$

where M_n is the mass, ω_n is the resonance frequency of the system, η_n is the damping loss factor (viscous damping has been assumed although inclusion of both viscous and solid type damping loss factors is possible). The point mobility of the system can be found as a summation of the response velocities at point x from each of the modes,

$$Y(x, \omega) = \sum_n \psi_n^2(x)\, Y_n(\omega), \tag{11b}$$

where $Y(x, \omega)$ is the point mobility of the system at point x and frequency ω and $Y_n(\omega)$ is the mobility function for the nth mode.

An average mobility over frequency or over an ensemble of systems can be found as the average of the individual terms of the summation,

$$\langle Y(x, \omega) \rangle = \sum_n \langle \psi_n^2(x)\, Y_n(\omega) \rangle, \tag{12a}$$

where $\langle\ \rangle$ represents an average. The average of the product can usually be expressed as the product of the averages. If a frequency average is being formed the mode shapes can be assumed to be frequency independent. If an ensemble average is being formed the mode shapes and the mode mobility function can be assumed to be statistically independent. In either case we can write the average mobility as

$$\langle Y(x, \omega) \rangle = \sum_n \langle \psi_n^2(x) \rangle \langle Y_n(\omega) \rangle. \tag{12b}$$

To evaluate the average we consider first the mobility functions for the individual modes, $Y_n(\omega)$.

For a lightly damped mode, the real part of the mobility shows a large peak at the resonance frequency. The imaginary part shows both a positive and negative peak near the resonance frequency. In evaluating the average mobility we consider first the conductance, the real part of Y. If we integrate the conductance of a single mode over a band of frequencies to determine the frequency-average, we can consider two distinct cases. When the resonance frequency is within the band, the integrated conductance will be large. In this case it is possible to extend the limits of integration to zero and infinity without significantly adding to the value of the integral. The resulting integral is simply $\pi/2M$. Thus, when the frequency band encompasses the resonance frequency of the mode, the frequency-average conductance for the mode is

$$\langle G_n \rangle_{\Delta\omega} = \frac{1}{\Delta\omega} \frac{\pi}{2M_n}, \tag{13}$$

where $\Delta\omega$ is the averaging bandwidth. We find that the average conductance does not depend on resonance frequency except to the extent that we require the

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frequency to be in the band. This leads to the idea that we do not need to know the exact resonance frequencies of a system, but only the number of modes with resonances in the band and the system mass. In addition, the average conductance does not depend on damping except to the extent that we require the damping to be small (damping loss factors less than 0.3 are sufficiently small that (13) gives a good estimate of the average conductance in the band). When the resonance frequency is outside the averaging band, the average conductance of the mode will be small and its contribution to the summation can generally be ignored.

The imaginary part of the mobility shows both a large positive and a large negative peak for the lightly damped system. At the resonance frequency the imaginary part of the mobility is zero. If we integrate over a frequency band containing the resonance frequency so that both the positive and negative peaks are included, the contribution to the two peaks tends to cancel so that the frequency-average of the imaginary part of the mobility tends to zero.

A frequency average mobility for the system is found by integrating the conductance of each mode over the bandwidth $\Delta\omega$ and summing over all modes. If we limit our attention to the conductance of lightly damped systems, we need consider only those modes with resonance frequencies in the band. For those modes we can extend the limits of integration to zero and infinity in evaluating the integral. The frequency-average point conductance can then be written as a simple summation over modes with resonance frequencies in the band

$$\langle G(x)\rangle_{\Delta\omega} = \frac{1}{\Delta\omega}\frac{\pi}{2}\sum_{\substack{\text{modes}\\ \text{in}\,\Delta\omega}}\frac{\psi_n^2(x)}{M_n}, \tag{14}$$

where $\Delta\omega$ is the averaging bandwidth and the summation is over all modes with resonance frequencies in the band.

Further simplification of the point conductance is possible, although not always justified. Simplification of the power input can result from averaging the conductance over an ensemble of systems. If the ensemble is defined so that the point at which the excitation is applied is a random variable the value of the mode shape squared can be replaced by an average value. For a homogeneous system this average value is simply the ratio of the modal mass divided by the physical mass of the system. The summation is then simply the number of modes with resonance frequencies in the band, $\Delta\omega$,

$$\langle G(x)\rangle_{\text{ens}} = \pi N/2M\,\Delta\omega, \tag{15}$$

where N is the mode count; the number of modes with resonance frequencies in the band. The ratio of the mode count to the bandwidth is the modal density for the system. Thus, we obtain the commonly used expression for the average conductance (Cremer 1988),

$$\langle G(x)\rangle_{\text{ens}} = \pi n(\omega)/2M, \tag{16}$$

where $n(\omega)$ is the modal density-average number of modes per unit radian frequency and M is the physical mass of the system.

The idea of a ensemble of systems with a randomly varying point of excitation is often hard to justify. However, we can also define an ensemble of systems with randomly varying boundary conditions. Because the mode shapes are strongly dependent on boundary conditions, we can achieve the same result for the ensemble average of the mode shape squared.

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The relation between the average conductance and the modal density is a key relation in SEA parameter derivatives. However, the assumptions used to derive the simple result in (16) from the more general result in (14) limit the usefulness of the simple result to homogeneous subsystems. Because all dynamic systems are inherently non-homogeneous, it is best to start with the general result in (14) and then determine if further simplification is possible. For example, if the SEA analysis is restricted to high frequencies and large frequency-averaging bandwidths, the excited SEA subsystems can be made sufficiently small that they can be modelled by homogeneous subsystems and (16) can be used. On the other hand, at low frequencies, or for narrow frequency bandwidths, small subsystems may have no modes. In this case it is necessary to increase the size of the excited subsystem and the more general result of (16) may be required.

(*d*) *Coupling loss factor*

In SEA the coupling loss factor relates the power transmitted between two connected subsystems to their energies

$$W_{\mathrm{s;r}}^{\mathrm{trans}} = \omega\eta_{\mathrm{s;r}}E_{\mathrm{s}} - \omega\eta_{\mathrm{r;s}}E_{\mathrm{r}}, \tag{17}$$

where $W_{\mathrm{s;r}}^{\mathrm{trans}}$ is the time-average power transmitted from subsystem s to subsystem r, $\eta_{\mathrm{s;r}}$ is the coupling loss factor between subsystem s and subsystem r, and E_{s} is the total energy of subsystem s. The coupling loss factors are not reciprocal so that $\eta_{\mathrm{s;r}} \neq \eta_{\mathrm{r;s}}$.

To formulate the coupling loss factor using mobility functions we use the power input relations derived in §2*b* together with a relation between the energy of the system and the blocked force or free velocity. If the two systems s and r are disconnected at the junction the resulting velocities of these systems can be considered to be the free velocities in formulating the power transferred between the two systems. In SEA the free velocities of system s and system r are assumed to be uncorrelated. With this assumption the net power transmitted between the systems can now be expressed using the power input formulation in terms of free velocities,

$$W_{\mathrm{s;r}}^{\mathrm{trans}} = \langle V_{\mathrm{fs}}^2\rangle \frac{G_{\mathrm{r}}}{|Y_{\mathrm{s}}+Y_{\mathrm{r}}|^2} - \langle V_{\mathrm{fr}}^2\rangle \frac{G_{\mathrm{s}}}{|Y_{\mathrm{s}}+Y_{\mathrm{r}}|^2}, \tag{18a}$$

where $\langle V_{\mathrm{fs}}^2\rangle$ is the mean-square free velocity of system s and $\langle V_{\mathrm{fr}}^2\rangle^2$ is the mean-square free velocity of system r. Similarly, the power transmitted between the two systems can be written in terms of the mean-square blocked forces,

$$W_{\mathrm{s;r}}^{\mathrm{trans}} = \langle F_{\mathrm{bs}}^2\rangle \frac{R_{\mathrm{r}}}{|Z_{\mathrm{s}}+Z_{\mathrm{r}}|^2} - \langle F_{\mathrm{br}}^2\rangle \frac{R_{\mathrm{s}}}{|Z_{\mathrm{s}}+Z_{\mathrm{r}}|^2}, \tag{18b}$$

where $\langle F_{\mathrm{bs}}^2\rangle$ is the mean-square blocked force of system s and $\langle F_{\mathrm{br}}^2\rangle$ is the mean-square blocked force of system r.

To determine the coupling loss factors it is necessary to simplify (18) by averaging over frequency or over an ensemble of systems. As discussed in §2*b*, it is common in most SEA formulations to use average mobility or impedance functions in (18) to evaluate the transmitted power.

The frequency-average or ensemble-average mean-square free velocities of the two systems are assumed to be proportional to the mean-square kinetic energies of the systems. Because the mean-square kinetic and potential energies of a system are

equal for resonant vibration, we can also assume the mean-square free velocities to be proportional to the total energies of the systems,

$$\langle V_{\mathrm{fs}}^2 \rangle = E_{\mathrm{s}}/M_{\mathrm{s}} \quad \text{and} \quad \langle V_{\mathrm{fr}}^2 \rangle = E_{\mathrm{r}}/M_{\mathrm{r}}, \tag{19}$$

where M_{s} and M_{r} are the masses of the two systems.

The process of disconnecting the two systems changes the response of the systems and the distribution of energy. We assume, however, that the relation between the mean-square free velocities and the system energies continues to be valid. With these assumptions the coupling loss factor can be written as

$$\eta_{\mathrm{s;r}} = \frac{1}{\omega M_{\mathrm{s}}} \frac{\langle G_{\mathrm{r}} \rangle}{|\langle Y_{\mathrm{s}} \rangle + \langle Y_{\mathrm{r}} \rangle|^2} \quad \text{and} \quad \eta_{\mathrm{r;s}} = \frac{1}{\omega M_{\mathrm{r}}} \frac{\langle G_{\mathrm{s}} \rangle}{|\langle Y_{\mathrm{s}} \rangle + \langle Y_{\mathrm{r}} \rangle|^2}, \tag{20}$$

where $\langle\ \rangle$ indicates that an average mobility function is being used. A symmetric coupling factor can be defined as the product,

$$\phi_{\mathrm{s;r}} = \omega \eta_{\mathrm{s;r}}\, n(\omega)_{\mathrm{s}}, \tag{21}$$

where $\phi_{\mathrm{s;r}}$ is the coupling factor and $n(\omega)$ is the modal density as a function of radian frequency. The coupling factor can be expressed in terms of the average mobility functions as

$$\phi_{\mathrm{s;r}} = \frac{n(\omega)_{\mathrm{s}}}{M_{\mathrm{s}}} \frac{\langle G_{\mathrm{r}} \rangle}{|\langle Y_{\mathrm{s}} \rangle + \langle Y_{\mathrm{r}} \rangle|^2}. \tag{22}$$

The ratio of the modal density to the system mass is proportional to the average conductance of the system. Thus, the coupling factor can be written as

$$\phi_{\mathrm{s;r}} = \frac{1}{2\pi} \frac{4\langle G_{\mathrm{s}} \rangle \langle G_{\mathrm{r}} \rangle}{|\langle Y_{\mathrm{s}} \rangle + \langle Y_{\mathrm{r}} \rangle|^2}. \tag{23a}$$

A similar derivation using the blocked force instead of the free velocity results in an impedance function formulation for the coupling factor

$$\phi_{\mathrm{s;r}} = \frac{1}{2\pi} \frac{4\langle R_{\mathrm{s}} \rangle \langle R_{\mathrm{r}} \rangle}{|\langle Z_{\mathrm{jnc}} \rangle|^2}, \tag{23b}$$

where $\langle Z_{\mathrm{jnc}} \rangle$ is the frequency-average or ensemble-average junction impedance; the sum of the average impedance of all systems connected to the junction.

The coupling factor given by (23*a*, *b*) can be used for one-, two- and three-dimensional systems coupled at a point with a single junction degree of freedom. For the special case of two one-dimensional systems, the mobility ratio in (23*a*) can be replaced by a power transmission coefficient; the ratio of the power transmitted to the receiving system to the power incident from the source system. Although coupling loss factors are often derived from power transmission coefficients, the mobility or impedance formulations given by (23*a*, *b*) are more general.

3. Extended interactions

(*a*) *Point source*

The excitation can be modelled by a point source if its extent is small compared with the wavelengths of vibration in the excited system. The most common example of a point source is a mechanical shaker used to excite a structure at a point.

For the general case of a point excitation of a structural system we must consider all six degrees of freedom; three translational or force degrees of freedom and three rotational or moment degrees of freedom. Because of the orthogonality of the force and moment components in the three axes, the power input can be written as

$$W_{\text{in}} = \tfrac{1}{2}\,\text{Re}\sum_i\sum_j F_i^*\,Y_{i;j}\,F_j, \tag{24a}$$

where the subscripts i and j range from 1 to 6 for the six degrees of freedom, F_i is a force or moment amplitude, $Y_{i;j}$ is the i,j term of the mobility matrix, and F_i^* is the complex conjugate of the force amplitude F_i. The terms of the mobility matrix are the velocity, V_i, divided by the force amplitude, F_j, where V_i and F_j refer to one of the translational and/or rotational degrees of freedom.

From the principle of reciprocity the mobility matrix is symmetric. With this condition the power input can be written in terms of the conductance matrix,

$$W_{\text{in}} = \tfrac{1}{2}\sum_i\sum_j F_i^*\,G_{i;j}\,F_j, \tag{24b}$$

where the imaginary terms cancel.

The cross-terms needed to evaluate the power input can be neglected in an ensemble-average, because both the cross-terms of the excitation and the cross-terms of the conductance matrix can be assumed to average to zero or to a sufficiently small value that they can be neglected. Although some error may be incurred it is generally acceptable in an engineering application to neglect the contribution of the cross-terms so that the power input for the general point force excitation can be written as a simple sum over the six degrees-of-freedom,

$$\langle W_{\text{in}}\rangle = \sum_i \langle F_i^2\rangle\,\langle G_i\rangle, \tag{25}$$

where $\langle F_i^2\rangle$ is the mean-square force (or moment) for the ith degree-of-freedom and $\langle G_i\rangle$ is the average of the ith diagonal term of the conductance matrix.

(*b*) *Line source*

The formulation of power input can be extended to a force excitation distributed over a line by expanding the force into a series of orthogonal functions, usually a complex Fourier series,

$$F(x) = \sum_i F(k_i)\,\mathrm{e}^{-jk_i x}, \tag{26}$$

where $F(k_i)$ is the Fourier amplitude for the component of the force at the wavenumber, k_i. The velocity is also expanded into a Fourier series, so that the intensity at point x (input power per unit length) can be determined from a double summation over Fourier amplitudes,

$$I_{\text{in}}(x) = \tfrac{1}{2}\,\text{Re}\sum_i\sum_j F(k_i)^*\,V(k_j)\,\mathrm{e}^{j(k_i-k_j)\,x}, \tag{27}$$

where $I_{\text{in}}(x)$ is the power intensity at point x. The input power is found by integrating over the length of the excitation. Because of the orthogonality of the Fourier functions, the cross-terms drop out so that the power input can be written in terms of a single summation over the Fourier amplitudes,

$$W_{\text{in}} = \tfrac{1}{2}L\,\text{Re}\sum_i F(k_i)^*\,V(k_i), \tag{28}$$

where L is the length of the distributed force excitation. The Fourier amplitude of the response velocity, $V(k_i)$, can be expressed as the matrix product of a mobility matrix and the Fourier amplitudes of the applied force. Because of reciprocity the mobility matrix relating the complex Fourier amplitudes of the response velocity and the excitation force must be symmetric. Then the complex mobility matrix can be replaced by the conductance matrix, which is real. The power input can now be written as

$$W_{\text{in}} = \tfrac{1}{2}L \sum_i \sum_j F(k_i)^* \, G(k_i, k_j) \, F(k_j), \tag{29}$$

where the imaginary parts of the matrix terms cancel. A term of the conductance matrix, $G(k_i, k_j)$, is the ratio of the complex Fourier amplitude of the response velocity at wavenumber k_i to the complex Fourier amplitude of the excitation force at wavenumber k_j.

The terms of the conductance matrix can be formulated in terms of the modes of the system as,

$$\langle G(k_i, k_j) \rangle_{\Delta\omega} = \frac{1}{\Delta\omega} \frac{\pi}{2} \sum_n \frac{\tilde{\psi}_n(k_i) \, \tilde{\psi}_n(k_j)^*}{M_n}, \tag{30a}$$

where the summation is over all modes with resonance frequencies in the band $\Delta\omega$ and the Fourier transform of the mode shape is defined as

$$\tilde{\psi}_n(k_i) = \frac{1}{L} \int_0^L \mathrm{d}x \, \psi_n(x) \, \mathrm{e}^{jk_i x}. \tag{30b}$$

In forming an ensemble average of the power input the cross terms of the conductance matrix can be neglected, because they can be assumed to average to zero. Thus the average power input from the line excitation can be written as

$$\langle W_{\text{in}} \rangle = L \sum_i \langle F(k_i)^2 \rangle \, \langle G(k_i) \rangle, \tag{29}$$

where $\langle F(k_i)^2 \rangle$ is the mean-square value of the Fourier component at k_i and $\langle G(k_i) \rangle$ is the average of a diagonal term of the conductance matrix.

(c) *Area source*

The power input from a pressure field associated with an acoustic source or a turbulent boundary layer is determined by a similar approach to that used of the line-distributed excitation. The area-distributed pressure field is expanded into Fourier amplitudes in two dimensions,

$$F(x, y) = \sum_{ix} \sum_{iy} F(k_{ix}, k_{iy}) \, \mathrm{e}^{-jk_{ix}x} \, \mathrm{e}^{-jk_{iy}y}, \tag{31}$$

where $F(k_{ix}, k_{iy})$ is the Fourier amplitude for the component of the force at the wavenumber defined by the components, k_{ix} and k_{iy}. The velocity is also expanded into its Fourier amplitudes and following the approach used for the line-distributed excitation the power input is found to be,

$$W_{\text{in}} = \tfrac{1}{2}A \sum_{ix} \sum_{iy} \sum_{jx} \sum_{jy} F(k_{ix}, k_{iy})^* \, G(k_{ix}, k_{iy}; k_{jx}, k_{jy}) \, F(k_{jx}, k_{jy}), \tag{32}$$

where the imaginary parts of the matrix terms cancel. The conductance matrix,

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$G(k_{ix}, k_{iy}; k_{jx}, k_{jy})$ is the real part of the ratio of the complex Fourier amplitude of the response velocity at the wavenumber defined by the components k_{ix} and k_{iy} to the complex amplitude of the excitation force at the wavenumber defined by the components k_{jx} and k_{jy}. As for the case of a line excitation the cross terms can be neglected in an ensemble average of the power input so that

$$\langle W_{\text{in}} \rangle = A \sum_{ix} \sum_{iy} \langle F(k_{ix}, k_{iy})^2 \rangle \langle G(k_{ix}, k_{iy}) \rangle, \tag{33}$$

where $\langle F(k_{ix}, k_{iy})^2 \rangle$ is the mean-square value of the Fourier component at k_{ix} and k_{iy} and $\langle G(k_{ix}, k_{iy}) \rangle$ is the average of a diagonal term of the conductance matrix.

4. Conclusions

The use of power and energy variables to describe the dynamic response of structural-acoustic systems is a key concept of SEA. In this paper the power input to a subsystem and the power transmitted between connected systems have been formulated using mobility functions. This in itself does not represent an advance to SEA theory. In fact, nearly all published derivations of SEA parameters are based, at least implicitly, on a mobility formulation. Because of the complexity of the general mobility formulation for most structural-acoustic problems few exact solutions for SEA parameters exist. The difficulty comes in identifying the approximations that are made and their appropriateness to the problems being considered. In this paper we have shown that a generalized and simplified formulation of the power input and coupling loss factor can be achieved by averaging either over frequency or over an ensemble of systems. The averaging process results in two general simplifications. First, it is usually possible to ignore the cross-terms in the mobility formulations. Second, it is possible to relate the conductance to the modal density and the mass of the system. A formal proof of the validity of these simplifications is not always possible. However, by making these simplifications in computing the SEA parameters complex dynamic problems that are otherwise intractable can be analysed using SEA.

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